

# Integral Representations for Elliptic Functions

Andrew Dienstfrey\* and Jingfang Huang†

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## Abstract

We derive new integral representations for objects arising in the classical theory of elliptic functions: the Eisenstein series  $E_s$ , and Weierstrass'  $\wp$  and  $\zeta$  functions. The derivations proceed from the Laplace-Mellin transformation for multipoles, and an elementary lemma on the summation of 2D geometric series. In addition, we present new results concerning the analytic continuation of the Eisenstein series as an entire function in  $s$ , and the value of the conditionally convergent series, denoted by  $\tilde{E}_2$  below, as a function of summation over increasingly large rectangles with arbitrary fixed aspect ratio<sup>1</sup>.

## 1 Introduction

In this paper we revisit the classical theory of elliptic functions as developed by Eisenstein and Weierstrass. Both of these researchers represented the meromorphic functions appearing in their theories as summations over a given lattice of elementary pole functions of a prescribed order. Our fundamental observation is that pole functions may be represented by exponentially-damped, oscillatory integrals. These representations depend on the complex half-planes in which the singularities lie, and are natural variants of the classical Mellin, or Laplace-Mellin, formulas which are valid for isolated poles lying in the right half plane, where  $\Re(\omega) > 0$ . In more recent times such integral representations have resurfaced in the development of fast multipole methods for the solution of Poisson problems; in this context they often are referred to as “plane-wave” representations ([6] and, more recently, [3]). A key feature of these integral representations is that the pole centers appear in the exponents of the integrands. As a consequence the lattice summations are transformed into geometric series which may be summed explicitly underneath the integral. The result is a new class of integral representations for the Eisenstein series and other meromorphic functions of Weierstrass' theory.

A brief summary of the paper follows. In the first section we review the definitions of the Eisenstein series  $E_n$  and the Weierstrass functions  $\wp$  and  $\zeta$ . We will analyze a generalization of Eisenstein's series which we denote by  $\tilde{E}_s$ , the differences being: first, we consider  $s = \sigma + it \in \mathbf{C}$ , and second, we define  $\tilde{E}_s$  as a limit over lattice squares of increasing size, a significant point when  $\Re(s) \leq 2$  and the sums are not absolutely convergent. In addition, in this preliminary section we provide elementary derivations of the requisite plane-wave formulas for general pole

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\*National Institute of Standards and Technology, Boulder, CO 80305-3328.

†University of North Carolina, Chapel Hill, NC 27599-6011.

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functions of the form  $f(\omega) = \omega^{-s}$ , and a summation identity for a two-dimensional geometric series.

In the next section we employ the plane-wave representations and the summation identity to derive an integral representation for  $\tilde{E}_s$  for the case  $\Re(s) > 2$ . Integral representations for Eisenstein's  $E_n$  naturally follow for  $s = n \geq 3$ . Subsequently, we derive an alternative representation for  $\tilde{E}_s$  as a contour integral from which we deduce that the sums  $\tilde{E}_s$ , defined unambiguously for  $\Re(s) > 2$ , admit an analytic continuation as an **entire** function to the whole of the complex plane. As a corollary, we prove the existence of a finite limit for  $\tilde{E}_2$ . We discuss  $\tilde{E}_2$  and its relation to Eisenstein's,  $E_2$ . As the summation processes defining these two conditionally convergent series are distinct, one expects different limiting values. We derive a closed form correction term for this difference. (Note that both Eisenstein's convention and ours give  $E_1 = \tilde{E}_1 = 0$ .) In the following section we derive analogous integral formulas for Weierstrass'  $\wp$  and  $\zeta$  functions. We conclude the paper with a brief discussion of these integral representations in relation to previous research in the theories of lattice sums, and elliptic functions.

In this last regard we take a moment to mention here that our formulas for  $\tilde{E}_s$  are the natural lattice analogues to the well-known representation for Riemann's zeta function (which we denote with the subscript  $\zeta_R$  so as to distinguish it from Weierstrass' function of the same name)

$$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \lambda^{s-1} \frac{1}{1 - e^{-\lambda}} e^{-\lambda} d\lambda, \quad \Re(s) > 1. \quad (1)$$

For example, in the case of a square lattice we derive the following integral expression for the classical Eisenstein series

$$\begin{aligned} E_k(i) &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(m + ni)^k} \\ &= \frac{8}{(k-1)!} \int_0^{\infty} \lambda^{k-1} \frac{\cos^2(\lambda/2)}{1 - 2e^{-\lambda} \cos(\lambda) + e^{-2\lambda}} e^{-\lambda} d\lambda, \end{aligned} \quad (2)$$

where  $k$  is a positive integer divisible by four ( $E_k(i) = 0$  otherwise). The similarity between (1) and (2) is clear. For more general lattices, we replace  $i$  by  $\tau$ ,  $k \in \mathbf{N}$  by  $s \in \mathbf{C}$ , and the single trigonometric ratio in (2) by a sum of analogous ratios denoted by  $f_1(\tau, \lambda)$  and  $f_2(\tau, \lambda)$  defined in (17) and (19). The general expression is given in theorem 4.

We note that a subset of the results presented below appeared previously in a slightly different form [7].

## 2 Preliminaries

In this section we review the definitions of the Eisenstein series and the Weierstrass  $\wp$  and  $\zeta$  functions. Furthermore, we derive elementary lemmas concerning plane-wave representations and a geometric series identity, both of which we will use repeatedly in the subsequent sections.

## 2.1 The Eisenstein series and Elliptic functions

We are given a general lattice  $\Lambda \subset \mathbf{C}$  which we describe by generators  $\mu, \nu$ :

$$\Lambda = \{m \cdot \mu + n \cdot \nu | m, n \in \mathbf{Z}\}$$

where  $\mu, \nu$  are complex numbers such that the lattice ratio,  $\tau = \nu/\mu$ , is not real. The fundamental parallelogram is the set,  $\Lambda_0 = \{\alpha + \beta\tau | |\alpha| \leq 1/2, |\beta| \leq 1/2\}$ . Eisenstein began his investigations of doubly periodic, meromorphic functions with periods  $\mu$  and  $\nu$  through the study his eponymous series (see, for example, [12] and [9])

$$E_k = \lim_{N \rightarrow \infty} \sum_{n=-N}^N \lim_{M \rightarrow \infty} \sum_{m=-M}^M \frac{1}{(m \cdot \mu + n \cdot \nu)^k}, \quad k \geq 1; \quad (3)$$

the elimination of the term  $m = n = 0$  is implicit here and below. For the magnitude of each term in the summand we have the bounds

$$\frac{k(\mu, \nu)}{\sqrt{m^2 + n^2}} \leq \frac{1}{|m \cdot \mu + n \cdot \nu|} \leq \frac{K(\mu, \nu)}{\sqrt{m^2 + n^2}}. \quad (4)$$

Applying (4) and elementary estimates one verifies that the series (3) are absolutely convergent for  $n \geq 3$ , and thus are well-defined functions of the lattice  $E_n = E_n(\Lambda)$ . The same estimates indicate that the series  $E_n$  are absolutely divergent for  $n = 1$  or  $2$ , hence the limiting operation specified in (3) plays a non-trivial role in the definition of these sums. Eisenstein proved that the limiting procedure (3) yields finite values of  $E_n$  even in these cases. We observe that, in particular,  $E_1 = 0$ . This is in keeping with the fact that, using a symmetry argument in the absolutely convergent case, one may prove that  $E_n = 0$  for all odd  $n \geq 3$ . Therefore, from the point of view of convergence, the only “interesting” sum is  $E_2$ .

Eisenstein was cognizant that the value of  $E_2$  depends on the choice of limiting procedures; and he derived many identities which connect his summation process for  $E_2$  to others ([12]). We choose yet a different summation convention and define  $\tilde{E}_s$  as the limit of partial sums over “lattice-squares” of increasing size. We generalize further in considering complex exponents. Specifically, we define  $\tilde{E}_s$  by

$$\tilde{E}_s = \lim_{K \rightarrow \infty} \sum_{|m|, |n| \leq K} \frac{1}{(m \cdot \mu + n \cdot \nu)^s}, \quad (5)$$

which we consider, initially, for  $s = 1, s = 2$  and  $\Re(s) > 2$ . For non-integer  $s$  we consider the branch of the function  $\zeta^s$  with a cut along the “negative diagonal” of the lattice,  $\{z = -t(\mu + \nu), t > 0\}$ . For all  $z$  in the closure of this cut plane we have

$$\begin{aligned} \theta &\leq \arg(z) \leq \theta + 2\pi, \text{ where} \\ \theta &= \text{Arg}(-\mu - \nu). \end{aligned} \quad (6)$$

We further enforce the convention that points of the lattice lying along the diagonal are considered symmetrically,

$$\frac{1}{(-m\mu - m\nu)^s} = \frac{1}{2(m|\mu + \nu|)^s} \left( \frac{1}{e^{i\theta s}} + \frac{1}{e^{i(\theta+2\pi)s}} \right)$$

We will return to this point later. Finally, we note that the large  $K$  limiting convention defined in (5) is relevant only in the cases  $s = 1$  or  $s = 2$ .

We will derive integral representations for  $\tilde{E}_s$ . Naturally, restricting  $s$  to the positive integers, our formula yields an integral representation for the classical Eisenstein series  $\tilde{E}_n = E_n$ , for  $n \geq 3$ . As for the conditionally convergent series, it is straightforward to verify that  $\tilde{E}_1 = 0$  directly from (5). However, for  $s = 2$ , it is not *a priori* obvious that (5) will be finite for arbitrary  $\Lambda(\mu, \nu)$ . Furthermore, if finite, the relationship between its value and Eisenstein's  $E_2$  is not clear. The finite existence of the limit (5) will follow as a consequence of the integral representations for  $\tilde{E}_s$ ,  $\Re(s) > 2$ . In addition, we derive a formula which connects our limiting value to Eisenstein's. Even more, we prove that  $\tilde{E}_s$  admits an analytic continuation to  $s \in \mathbb{C}$  as an entire function.

Some fifteen years on the heels of Eisenstein, in 1862 Weierstrass commenced his study of doubly-periodic functions. For Weierstrass, the fundamental object was his  $\wp$  function which he defined as

$$\wp(x, \Lambda) = \frac{1}{x^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left( \frac{1}{(x - \omega)^2} - \frac{1}{\omega^2} \right). \quad (7)$$

Standard estimates demonstrate that  $\wp$  is an analytic function of  $x$ , and well-defined function of  $\Lambda$ . From (7) we see that the  $\wp$  function has a double pole at the origin and every lattice translate. Weierstrass defined his  $\zeta$  function as an indefinite integral of  $\wp$  and developed the following summation representation

$$\zeta(x, \Lambda) = - \int^x \wp(s, \Lambda) ds = \frac{1}{x} + \sum_{\omega \in \Lambda \setminus \{0\}} \left( \frac{1}{(x - \omega)} + \frac{1}{\omega} + \frac{x}{\omega^2} \right). \quad (8)$$

Again, the correction terms inside the summation balance the asymptotics of the translated poles so that the sum converges absolutely for  $x$  in a compact set containing no lattice points. As  $\zeta$  is meromorphic and has only a single pole within a fundamental parallelogram, it can not be doubly periodic (see, for example, [1]). Nevertheless, Weierstrass derived many properties satisfied by this function and its relatives. The absolutely convergent sums (7) and (8) will serve as the starting points for the derivation of the integral representations for  $\wp$  and  $\zeta$  below.

We conclude this section with a brief comment on freedom in the choice of generators for  $\Lambda$ . Note that  $\tilde{E}_s, \wp, \zeta$  satisfy simple rescalings with respect to  $\mu$

$$\begin{cases} \tilde{E}_s(\mu, \nu) = \frac{1}{\mu^s} \tilde{E}_s(1, \tau), \\ \wp(x|\mu, \nu) = \frac{1}{\mu^2} \wp\left(\frac{x}{\mu} \middle| 1, \tau\right), \\ \zeta(x|\mu, \nu) = \frac{1}{\mu} \zeta\left(\frac{x}{\mu} \middle| 1, \tau\right), \end{cases} \quad (9)$$

where  $\tau = \nu/\mu$ .

Furthermore, one observes that two pairs of generators  $(\mu, \nu)$  and  $(\mu', \nu')$  will give rise to the same lattice if and only if they satisfy a linear system of the form

$$\begin{pmatrix} \mu' \\ \nu' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix}$$

with  $a, b, c, d \in \mathbf{Z}, ad - bc = \pm 1$ . The set of all two-by-two matrices with integral entries and determinant plus or minus one is known as the unimodular group in two variables. In this case the lattice ratios are transformed by the fractional linear transformation

$$\tau' = \frac{a + b\tau}{c + d\tau},$$

known as a unimodular substitution. One may prove that up to rescaling and unimodular substitution, any lattice ratio may be represented by a unique  $\tau$  chosen from the following fundamental region [1]

$$\begin{cases} -\frac{1}{2} < \Re(\tau) \leq \frac{1}{2}, \\ \Im(\tau) > 0, \\ |\tau| \geq 1, \\ \text{if } |\tau| = 1, \text{ then } \Re(\tau) \geq 0. \end{cases} \quad (10)$$

In summary, without loss of generality, we restrict our analysis to the “inhomogeneous” functions, which are obtained from (5), (7) and (8) by fixing  $\mu = 1, \nu = \tau$ , and consider  $\Lambda = \Lambda(\tau)$  with  $\tau$  satisfying (10). For convenience we omit the variables  $\mu, \nu$  below and write, for example,  $\wp = \wp(z, \tau)$ .

## 2.2 Plane-wave representations and a 2D Geometric Series

To facilitate our derivations we define the truncated lattice  $\Lambda^K = \{\omega_{m,n} = m + n\tau \mid |m|, |n| < K\} \setminus \{0\}$ . We further group lattice points into four overlapping “quadrants”

$$\Lambda^K = \Lambda_{(+,\bullet)}^K \cup \Lambda_{(\bullet,+) }^K \cup \Lambda_{(-,\bullet)}^K \cup \Lambda_{(\bullet,-)}^K$$

defined by

$$\begin{aligned} \Lambda_{(+,\bullet)}^K &= \{\omega_{m,n} \mid 1 \leq m \leq K, |n| \leq m\} \\ \Lambda_{(\bullet,+) }^K &= \{\omega_{m,n} \mid 1 \leq n \leq K, |m| \leq n\} \\ \Lambda_{(-,\bullet)}^K &= \{\omega_{m,n} \mid -K \leq m \leq -1, |n| \leq -m\} \\ \Lambda_{(\bullet,-)}^K &= \{\omega_{m,n} \mid -K \leq n \leq -1, |m| \leq -n\}. \end{aligned} \quad (11)$$

We recall from the discussion following (5) that for non-integer  $s$  the shared boundary between  $\Lambda_{(-,\bullet)}^K$  and  $\Lambda_{(\bullet,-)}^K$  is identical to the branch cut (see figure 5).

We have the following elementary lemma

**Lemma 1.** *Assume a complex lattice  $\Lambda(\tau)$  and the quadrants defined as in (11). An isolated singularity of complex order  $s$ ,  $\Re(s) > 0$  with branch cut defined as in (6) may be represented by the following plane-wave integrals, each of which is valid in the appropriate quadrant determined*

by the location of the point  $\omega$ .

$$\frac{1}{\omega^s} = \begin{cases} \frac{1}{\Gamma(s)} \int_0^\infty \lambda^{s-1} e^{-\lambda\omega} d\lambda, & \omega \in \Lambda_{(+,\bullet)}^K \\ \frac{e^{-i\pi s}}{\Gamma(s)} \int_0^\infty \lambda^{s-1} e^{\lambda\omega} d\lambda, & \omega \in \Lambda_{(-,\bullet)}^K \\ \frac{e^{-i\pi s/2}}{\Gamma(s)} \int_0^\infty \lambda^{s-1} e^{i\lambda\omega} d\lambda, & \omega \in \Lambda_{(\bullet,+)}^K \\ \frac{e^{i\pi s/2}}{\Gamma(s)} \int_0^\infty \lambda^{s-1} e^{-i\lambda\omega} d\lambda, & \omega \in \Lambda_{(\bullet,-)}^K. \end{cases} \quad (12)$$

**Proof:**

We have the representation of the  $\Gamma$  function

$$\Gamma(s) = \int_0^\infty \lambda^{s-1} e^{-\lambda} d\lambda, \quad \Re(s) > 0.$$

Assume  $\omega \in i\mathbf{R}^+$ , in particular,  $\omega = it$ ,  $t > 0$ . As  $\tau$  satisfies (10) we observe that  $-\pi < \theta < -\pi/2$  hence  $\arg(\omega^s) = \pi s/2$ . Keeping this in mind, we rescale the integration variable by  $t$ , factor the  $-1$  in the exponential, and multiply and divide by  $\exp(i\pi s/2)$  to obtain

$$\begin{aligned} \Gamma(s) &= t^s \int_0^\infty \lambda^{s-1} e^{-\lambda t} d\lambda \\ &= e^{-i\pi s/2} \omega^s \int_0^\infty \lambda^{s-1} e^{i\lambda\omega} d\lambda. \end{aligned}$$

Dividing both sides by  $\Gamma(s)\omega^s$  gives the desired result for  $\omega = it$ . In a similar manner we prove the formula for  $\omega$  lying on any of the principal coordinate rays emanating from the origin,  $\omega \in \pm\mathbf{R}^+, \pm i\mathbf{R}^+$ . The full expressions (12) then follow by analytic continuation into the appropriate quadrants.  $\dagger$

We note that for integer  $s$ , the integral expressions may be continued further and are valid in the appropriate half-planes  $\pm\Re(\omega) > 0$  and  $\pm\Im(\omega) > 0$ .

Next we turn to our summation convention (5). From lemma 1, it is apparent that no single plane-wave expansion formula will be valid for all terms in the summands (5), (7), and (8); terms must be grouped with respect to quadrant. As with the our convention of splitting contributions from points  $\omega_{-m,-m}$  lying on the cut in (5) equally between branches, we wish to treat each quadrant as symmetrically as possible. We define the symbol  $\epsilon_{mn}$  for  $m, n \in \mathbf{Z}$  by

$$\epsilon_{mn} = \begin{cases} \frac{1}{2}, & m = \pm n \\ 1, & \text{otherwise} \end{cases} \quad (13)$$

By convention, we sum over the terms in the  $\Lambda_{(+,\bullet)}^K$ -quadrant as

$$\sum_{\omega \in \Lambda_{(+,\bullet)}^K} f(\omega) = \sum_{m=1}^K \sum_{n=-m}^m \epsilon_{mn} f(\omega_{m,n}) \quad (14)$$

$$= \sum_{m=1}^K \left( \frac{1}{2} f(\omega_{m,-m}) + \sum_{n=-m+1}^{m-1} f(\omega_{m,n}) + \frac{1}{2} f(\omega_{m,m}) \right) \quad (15)$$

We recognize the slight abuse of notation in (14) in that the left hand side is not interpreted as a standard sum over the set  $\Lambda_{(+,\bullet)}^K$ , but rather a modified sum in which the “diagonal” terms are added with a factor of  $1/2$ ; this is made explicit in (15). We choose this abuse of notation so as to not burden our summation symbols with a cascade of modifiers, and anticipate that it will not cause confusion below.

The sum over the lattice square  $\Lambda^K$  is the sum of the quadrant sums as in (14); hence the reason for the factor of  $1/2$  — to avoid double-counting of the contributions from the diagonal terms — is clear. We note that for  $s \in \mathbf{N}$ ,  $s > 2$  the numerical value of the sums is independent of any manner of grouping terms. Even so, the form of the integrands in our integral representations reflect this choice. We have found that the convention (14) yields the most symmetric expressions in appearance (a different grouping for integer  $s$  was employed in [7]).

Next, we recognize that in sums of the form (14) we may substitute the appropriate plane-wave expansion (12) to represent the poles contained in  $f$ . This substitution will transform the quadrant sums into geometric series. Concerning the later, we derive the following lemma.

**Lemma 2.** *For any  $p, q \in \mathbf{C}$  and  $K \in \mathbf{N}$ , the following is an identity*

$$\sum_{i=1}^K \sum_{j=-i}^i \epsilon_{ij} p^i q^j = \frac{1}{2} \frac{p(q^{-1} + 2 + q)}{(1 - p(q + q^{-1}) + p^2)} - \frac{1}{2} \left( \frac{1+q}{1-q} \right) \left[ \frac{(pq^{-1})^{K+1}}{1 - pq^{-1}} - \frac{(pq)^{K+1}}{1 - pq} \right]. \quad (16)$$

**Proof:**

The formula follows from iteration of the usual single variable geometric sum, and algebra.

We record the following corollary for reference as the expressions appear many times in the formulas below.

**Corollary 3.** *We have the following specializations of lemma 2*

$$\sum_{m=1}^K \sum_{n=-m}^m \epsilon_{mn} (e^{-\lambda})^m (e^{-\lambda\tau})^n = 2e^{-\lambda} f_1(\tau, \lambda) - 2e^{-\lambda(K+1)} f_1^{(K)}(\tau, \lambda)$$

and

$$\sum_{n=1}^K \sum_{m=-n}^n \epsilon_{nm} (e^{i\tau\lambda})^n (e^{i\lambda})^m = 2e^{i\tau\lambda} f_2(\tau, \lambda) - 2e^{i\tau\lambda(K+1)} f_2^{(K)}(\tau, \lambda)$$

where the functions  $f_1, f_1^{(K)}, f_2, f_2^{(K)}$  are:

$$f_1(\tau, \lambda) = \frac{\cosh^2(\tau\lambda/2)}{1 - 2e^{-\lambda} \cosh(\tau\lambda) + e^{-2\lambda}} \quad (17)$$

$$f_1^{(K)}(\tau, \lambda) = \frac{1}{4} \left( \frac{1 + e^{-\lambda\tau}}{1 - e^{-\lambda\tau}} \right) \left[ \frac{e^{\lambda\tau(K+1)}}{1 - e^{-\lambda(1-\tau)}} - \frac{e^{-\lambda\tau(K+1)}}{1 - e^{-\lambda(1+\tau)}} \right] \quad (18)$$

$$f_2(\tau, \lambda) = \frac{\cos^2(\lambda/2)}{1 - 2e^{i\tau\lambda} \cos(\lambda) + e^{2i\tau\lambda}} \quad (19)$$

$$f_2^{(K)}(\tau, \lambda) = \frac{1}{4} \left( \frac{1 + e^{i\lambda}}{1 - e^{i\lambda}} \right) \left[ \frac{e^{-i\lambda(K+1)}}{1 - e^{i\lambda(\tau-1)}} - \frac{e^{i\lambda(K+1)}}{1 - e^{i\lambda(\tau+1)}} \right]. \quad (20)$$

We assume  $\tau$  is in the fundamental region (10) and make several observations. First, we note that all of the functions given by (17)-(20) have double poles at the origin,  $\lambda = 0$ . Since  $\Im(\tau) > 0$ , neither  $f_1(\tau, \lambda)$  nor  $f_2(\tau, \lambda)$  have other poles for  $\lambda > 0$ . For  $\tau$  strictly imaginary, the denominator  $1 - \exp(-\tau\lambda)$  of  $f_1^{(K)}(\tau, \lambda)$  will have isolated simple zeros. However, these are balanced by simple zeros of the difference of bracketed terms in (18). Thus the product, in other words  $f_1^{(K)}$ , has no other singularities for  $\lambda > 0$ . A similar argument shows that  $f_2^{(K)}$  is also finite for  $\lambda > 0$ . Finally, one may show the bounds

$$|e^{-\lambda} f_1(\tau, \lambda)| < C_1 e^{-\lambda(1-|\Re(\tau)|)}, \text{ and} \\ |e^{i\tau\lambda} f_2(\tau, \lambda)| < C_2 e^{i\tau\lambda}$$

for large  $\lambda$ . As  $|\Re(\tau)| \leq 1/2$  and  $\Im(\tau) > 0$ , both quantities are exponentially decreasing in  $\lambda$ . Similar reasoning shows that  $e^{-\lambda(K+1)} f_1^{(K)}(\tau, \lambda)$  and  $e^{i\tau\lambda(K+1)} f_2^{(K)}(\tau, \lambda)$  are exponentially decreasing in  $\lambda$  and  $K$ .

### 3 Eisenstein series

As mentioned previously, the summation  $\tilde{E}_s$  for  $\Re(s) > 2$  is absolutely convergent. We begin by proving our first integral representation for this case in theorem 4. As a corollary, by restricting  $s = n$ ,  $n \geq 3$  we obtain integral representations of  $E_n$ . Further inspection of the integral representation demonstrates the existence of  $\tilde{E}_2$ . Elaborating on theorem 4, we derive an alternative representation for  $\tilde{E}_s$  as a contour integral. As a consequence of this second representation, we prove that  $\tilde{E}_s$  admits an analytic continuation in  $s$  as an entire function. Returning to the analysis of  $\tilde{E}_2$ , we consider a more general limiting procedure and define  $\tilde{E}_2^{(\alpha)}$  as the limit over increasing “lattice rectangles” with a fixed aspect ratio defined by  $\alpha$ . We derive a closed form expression which, when added to  $\tilde{E}_2 = \tilde{E}_2^{(1)}$ , gives  $\tilde{E}_2^{(\alpha)}$ . As a corollary, we derive the relationship between  $\tilde{E}_2$  and the sum  $E_2$  as defined by Eisenstein.

#### 3.1 Integral representations

For the sums  $\tilde{E}_s$  defined by (5) we prove



**Theorem 4.** Given a lattice  $\Lambda(\tau)$  with ratio  $\tau$  chosen from the fundamental region (10), we have the following integral representation for  $\tilde{E}_s, \Re(s) > 2$

$$\tilde{E}_s(\tau) = \cos\left(\frac{\pi}{2}s\right) \frac{4}{\Gamma(s)} \int_0^\infty \lambda^{s-1} \left( e^{-is\pi/2} e^{-\lambda} f_1(\tau, \lambda) + e^{i\tau\lambda} f_2(\tau, \lambda) \right) d\lambda \quad (21)$$

where  $f_1(\tau, \lambda)$  and  $f_2(\tau, \lambda)$  are given by (17) and (19).

**Proof:**

We recognize that, due to the placement of branch cut (6) and the symmetric summation conventions (14), we have the following relations between sums over  $\Lambda_{(\pm, \bullet)}^K$  and  $\Lambda_{(\bullet, \pm)}^K$

$$\begin{aligned} \sum_{\omega \in \Lambda_{(-, \bullet)}^K} \frac{1}{\omega^s} &= e^{-is\pi} \sum_{\omega \in \Lambda_{(+, \bullet)}^K} \frac{1}{\omega^s} \\ \sum_{\omega \in \Lambda_{(\bullet, -)}^K} \frac{1}{\omega^s} &= e^{is\pi} \sum_{\omega \in \Lambda_{(\bullet, +)}^K} \frac{1}{\omega^s}. \end{aligned}$$

Therefore we may consider the positive quadrants only and scale the results by an exponential factor. Turning to the quadrant  $\Lambda_{(+, \bullet)}^K$ , in place of the isolated singularity of degree  $s$ , we substitute the appropriate plane-wave expression from (12) to obtain

$$\begin{aligned} \sum_{\omega \in \Lambda_{(\pm, \bullet)}^K} \frac{1}{(m + n\tau)^s} &= (1 + e^{-is\pi}) \sum_{m=1}^K \sum_{n=-m}^m \epsilon_{mn} \frac{1}{(m + n\tau)^s} \\ &= \frac{(1 + e^{-is\pi})}{\Gamma(s)} \int_0^\infty \lambda^{s-1} \sum_{m=1}^K \sum_{n=-m}^m \epsilon_{mn} e^{-\lambda(m+n\tau)} d\lambda \\ &= \frac{2(1 + e^{-is\pi})}{\Gamma(s)} \int_0^\infty \lambda^{s-1} \left( e^{-\lambda} f_1(\tau, \lambda) - e^{-\lambda(K+1)} f_1^{(K)}(\tau, \lambda) \right) d\lambda \end{aligned}$$

where the last line follows from corollary 3. From the statements following this same corollary, we observe that the two integrands are singular at  $\lambda = 0$ , and are otherwise finite and exponentially decreasing in  $K$  and  $\lambda \in \mathbf{R}^+$ . In addition, as  $\Re(s) > 2$ , the singularity at the origin is absolutely integrable. Therefore, one may take the large  $K$  limit inside the integral and compute

$$\lim_{K \rightarrow \infty} \sum_{\omega \in \Lambda_{(\pm, \bullet)}^K} \frac{1}{(m + n\tau)^s} = \cos\left(\frac{\pi}{2}s\right) \frac{4}{\Gamma(s)} \int_0^\infty \lambda^{s-1} e^{-is\pi/2} e^{-\lambda} f_1(\tau, \lambda) d\lambda.$$

By a similar analysis we prove that

$$\lim_{K \rightarrow \infty} \sum_{\omega \in \Lambda_{(\bullet, \pm)}^K} \frac{1}{(m + n\tau)^s} = \cos\left(\frac{\pi}{2}s\right) \frac{4}{\Gamma(s)} \int_0^\infty \lambda^{s-1} e^{i\tau\lambda} f_2(\tau, \lambda) d\lambda.$$

Adding these two contributions gives the theorem.  $\dagger$

A few comments are in order. First, for  $s = 2j + 1$  the term  $\cos(\frac{\pi}{2}s) = 0$ . This is equivalent to the well-known fact that, by symmetry, the odd Eisenstein series are zero. Furthermore, by inspection of (21), we see that the absolute convergence of the Eisenstein series  $\tilde{E}_s, \Re(s) > 2$  manifests itself in the behavior of the integrand of (21) near the origin; the factor  $\lambda^{s-1}$  balances the double poles of  $f_1$  and  $f_2$  so as to ensure the product is integrable at  $\lambda = 0$ . More careful analysis reveals that the formula (21) is finite even for the conditionally convergent case  $\tilde{E}_2$ . The Laurent expansions of the integrands about the origin are

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \lambda e^{-\lambda} \frac{\cosh^2(\tau\lambda/2)}{1 - 2e^{-\lambda} \cosh(\tau\lambda) + e^{-2\lambda}} &= \frac{1}{\lambda(1 - \tau^2)} - \frac{(1 - 2\tau^2)\lambda}{12(1 - \tau^2)} + O(\lambda^3) \\ \lim_{\lambda \rightarrow 0} \lambda e^{i\lambda\tau} \frac{\cos^2(\tau\lambda/2)}{1 - 2e^{i\tau\lambda} \cos(\tau\lambda) + e^{i2\tau\lambda}} &= \frac{1}{\lambda(1 - \tau^2)} - \frac{(2 - \tau^2)\lambda}{12(1 - \tau^2)} + O(\lambda^3) \end{aligned}$$

At  $s = 2$ , where the exponential factor  $e^{-is\pi/2} = -1$ , the expansions above are to be subtracted. Hence the integrand of (21) is finite at the origin even in this case. By a similar analysis one may show that the  $K$ -dependent terms also cancel at the origin. We have proved:

**Corollary 5.** *The summation (5) converges in the conditionally convergent case  $s = 2$ , and its value,  $\tilde{E}_2$ , is given by the integral (21).*

In fact, a great deal more may be said. The function

$$F(s, z) = e^{-is\pi/2} e^{-z} \frac{\cosh^2(\tau z/2)}{1 - 2e^{-z} \cosh(\tau z) + e^{-2z}} + e^{i\tau z} \frac{\cos^2(z/2)}{1 - 2e^{i\tau z} \cos(z) + e^{i2\tau z}} \quad (22)$$

appearing as a factor in the integrand (21) has a singularity at the origin  $z = 0$ , and additional simple poles in the complex plane at the points

$$z \in P = \left\{ \pm \frac{2\pi i}{1 \pm \tau} m, \pm \frac{2\pi}{1 \pm \tau} n \mid m, n \in \mathbf{N} \right\}.$$

We denote the minimum magnitude of all  $z \in P$  by  $\rho$ . Next, define the contour  $C$  which begins at  $\infty + iy$ ,  $y > 0$ ; runs parallel to real axis until it intersects the circle centered at the origin with radius  $r$ , where  $y < r < \rho$ ; follows this circle counterclockwise around the origin; and runs back out to  $\infty - iy$ , parallel to the real axis. We assume that  $y > 0$  is small enough such that  $C$  encloses only the pole at  $z = 0$ . Finally, for the function  $z^{s-1}, s \notin \mathbf{Z}$ , situate the branch cut along the positive real axis such that  $(\lambda \in \mathbf{R})$

$$\begin{cases} \lim_{y \rightarrow 0} (\lambda + iy)^{s-1} = \lambda^{s-1} \\ \lim_{y \rightarrow 0} (\lambda - iy)^{s-1} = e^{2\pi i(s-1)} \lambda^{s-1} \end{cases} \quad (23)$$

With these preliminaries established we prove the following theorem.

**Theorem 6.** *Given a lattice  $\Lambda(\tau)$  with ratio  $\tau$  chosen from the fundamental region (10), we have the following contour integral representation for  $\tilde{E}_s, \Re(s) > 2$*

$$\tilde{E}_s = 2 \cos\left(\frac{\pi}{2}s\right) \frac{\Gamma(1-s)e^{-is\pi}}{i\pi} \int_C z^{s-1} F(s, z) dz \quad (24)$$

where  $F(s, z)$  is given by (22).

**Proof:**

Consider the contour integral

$$\int_C z^{s-1} F(s, z) dz.$$

As the integrand is analytic except for the singularity at the origin, we may deform the contour without altering the value of the integral. Specifically, shrink the radius of the circle,  $r \rightarrow 0$ , and take the limit as  $y \rightarrow 0$  for the two components running parallel to the real axis. For  $\Re(s) = 2 + \epsilon$ , we estimate the contribution from the circular arc

$$\lim_{r \rightarrow 0} \left| \int_{|z|=r} z^{s-1} F(s, z) dz \right| \leq M \lim_{r \rightarrow 0} \int_{|z|=r} r^{\epsilon-1} |dz| = 0.$$

Turning to the components parallel to the real axis, applying the specification of the branch cut (23) we compute

$$\begin{aligned} \lim_{y \rightarrow 0} \int_{\infty+iy}^{\epsilon_r+iy} z^{s-1} F(s, z) dz &= - \int_0^\infty \lambda^{s-1} F(s, \lambda) d\lambda \\ \lim_{y \rightarrow 0} \int_{\epsilon_r-iy}^{\infty-iy} z^{s-1} F(s, z) dz &= e^{2\pi i(s-1)} \int_0^\infty \lambda^{s-1} F(s, \lambda) d\lambda. \end{aligned}$$

We recognize the integrals of theorem 4, make use of the identity

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)},$$

and obtain

$$\begin{aligned} \int_C z^{s-1} F(s, z) dz &= (e^{2\pi i(s-1)} - 1) \int_0^\infty \lambda^{s-1} F(s, \lambda) d\lambda \\ &= \frac{2ie^{\pi is} \sin(\pi s) \Gamma(s)}{4 \cos\left(\frac{\pi}{2}s\right)} \tilde{E}_s \\ &= \frac{i\pi}{2 \cos\left(\frac{\pi}{2}s\right) e^{-is\pi} \Gamma(1-s)} \tilde{E}_s. \end{aligned}$$

The result (24) follows from algebra.  $\dagger$

Several corollaries follow from theorem 6. We note here only the most immediate

**Corollary 7.** *The sums  $\tilde{E}_s$ , defined by (5) for  $\Re(s) > 2$ , admit an analytic continuation to  $s \in \mathbf{C}$  as an entire function. This continuation is given by the contour integral representation (24).*

**Proof:**

The contour integral appearing in (24) defines an analytic function of  $s$  which is never singular. The same may be said for the cosine factor. Thus the only possible singularities would arise from the factor  $\Gamma(1-s)$  which has simple poles for  $s \in \mathbf{N}$ . From the definition (5) we know that  $\tilde{E}_s$  is finite for  $\Re(s) > 2$  (the apparent singularities in (24) in this case are balanced by zeros of the cosine term, the contour integral, or both.) Therefore, we need only verify the finite existence of  $\tilde{E}_1$  and  $\tilde{E}_2$ . We have argued above that  $\tilde{E}_2$  is finite. Finally, for  $s = n = 1$ , the pole in the Gamma function is balanced by the simple zero of the cosine factor.  $\dagger$

By inspection of (24), we find that  $\tilde{E}_s$  has simple zeros for  $s = 1 - 2j$ ,  $j \geq 1$ . Thus the continuation respects the symmetric limiting process (5). We anticipate further results concerning the evaluation of the contour integral (24) via residue methods. However, we have not carried out this analysis at the time of writing this paper. Finally, as mentioned previously, the exact form of the integrands (21) and (24) reflect our summation convention with respect to grouping of summands and placement of the branch-cut. Regarding the latter, similar formulas arise if the branch-cut is situated along any of the lattice diagonals; the effect is to redistribute factors of  $\exp(i\pi s/2)$  between the two functions  $f_1$  and  $f_2$ . There are, perhaps, additional treatments of the branch-cut that could yield relatively simple expressions. However, a simple integral expression valid for placement of the cut along an arbitrary ray in the complex plane appears to be intractable.

### 3.2 Aspect ratio correction

Although we have demonstrated that  $\tilde{E}_2$  is finite, until this point its relation to Eisenstein's definition of this series is unclear. We make this explicit in the present section.

We wish to formalize summation over lattice rectangles with a fixed aspect ratio. Given  $\alpha \in (0, \infty)$  we define the sum

$$\tilde{E}_2^{(\alpha)} = \lim_{K \rightarrow \infty} \sum_{|m| \leq \lfloor \alpha K \rfloor} \sum_{|n| \leq K} \frac{1}{(m + n\tau)^2}, \quad (25)$$

where  $\lfloor x \rfloor$  denotes greatest integer less than or equal to  $x$ . Again, although expected, the existence of the limit (25) is not *a priori* guaranteed but will follow in the course of our analysis.

Clearly,  $\tilde{E}_2^{(1)} = \tilde{E}_2$  defined in (5). More generally, we write

$$\tilde{E}_2^{(\alpha)} = \tilde{E}_2 + \Delta(\alpha, \tau) \quad (26)$$

where the value of  $\tilde{E}_2$  may be computed via the integral expression (21). Analysis similar to that used to prove (21) may be employed to compute a closed form expression for  $\Delta(\alpha, \tau)$ .

**Theorem 8.** *For a fixed aspect ratio determined by  $\alpha \in (0, \infty)$ , the limit  $\tilde{E}_2^{(\alpha)}$  specified in (25) exists. Furthermore, when written in the form (26), the aspect ratio dependence is given by*

$$\Delta(\alpha, \tau) = -\frac{4i}{\tau} \left( \arctan(i\tau) - \arctan\left(\frac{i\tau}{\alpha}\right) \right) \quad (27)$$

**Proof:**

Assume  $\alpha \geq 1$ . We write the limit (25)

$$\begin{aligned}\tilde{E}_2^{(\alpha)} &= \tilde{E}_2 + \Delta(\alpha, \tau), \\ \Delta(\alpha, \tau) &= 2 \lim_{K \rightarrow \infty} \sum_{m=K+1}^{\lfloor \alpha K \rfloor} \sum_{n=-K}^{n=K} \frac{1}{(m+n\tau)^2}.\end{aligned}$$

The contribution from the sum over lattice points  $-\lfloor \alpha \cdot K \rfloor \leq m \leq -K-1$  is accounted for by the factor of two multiplying the sum in the final line. We represent the poles using the  $\Lambda_{(+,\bullet)}^K$  plane-wave expansion (12).

$$\begin{aligned}\Delta(\alpha, \tau) &= 2 \lim_{K \rightarrow \infty} \sum_{m=K+1}^{\lfloor \alpha K \rfloor} \sum_{n=-K}^{n=K} \int_0^\infty \lambda e^{-\lambda(m+n\tau)} d\lambda \\ &= 2 \lim_{K \rightarrow \infty} \int_0^\infty \lambda \left( \frac{e^{\lambda\tau K} - e^{-\lambda\tau(K+1)}}{1 - e^{-\lambda\tau}} \right) \left( \frac{e^{-\lambda(K+1)} - e^{-\lambda\lfloor \alpha K \rfloor}}{1 - e^{-\lambda}} \right) d\lambda \\ &= 2 \lim_{K \rightarrow \infty} \int_0^\infty \lambda \left( \frac{e^{\lambda\tau} - e^{-\lambda\tau(1+1/K)}}{1 - e^{-\lambda\tau/K}} \right) \left( \frac{e^{-\lambda(1+1/K)} - e^{-\lambda\lfloor \alpha K \rfloor/K}}{1 - e^{-\lambda/K}} \right) \frac{1}{K^2} d\lambda \\ &= \frac{2}{\tau} \int_0^\infty \frac{(e^{\lambda\tau} - e^{-\lambda\tau})(e^{-\lambda} - e^{-\lambda\alpha})}{\lambda} d\lambda.\end{aligned}\tag{28}$$

Where the argument which justifies taking the large  $K$  limit inside the integral runs along the same lines as in the proof of theorem 4. This last integral (28) may be evaluated in closed form using the formula

$$\int_0^\infty e^{-\beta x} \sin(\delta x) \frac{1}{x} dx = \arctan\left(\frac{\delta}{\beta}\right),\tag{29}$$

which holds for  $\Re(\beta) > |\Im(\delta)|$  (see [4], 3.944.5). Taking care to write (28) as the difference of two integrals of the form (29), and performing algebra gives the expression for  $\Delta(\alpha, \tau)$ ,  $\alpha \geq 1$  in (27).

For  $\alpha < 1$  the lattice rectangle is such that the longer side is in the  $\tau$ -direction. As written, equations (25) and (26) suggest that this rectangle is inscribed in a lattice square of size  $K$ , and to compute  $\Delta(\alpha, \tau)$ , one should subtract the extra contributions exterior to the rectangle but interior to the square. The problem with this approach is that, for arbitrary  $\tau$  and  $\alpha$ , one would have to keep track of the quadrants in which these points lie. In lieu of this, for  $\alpha < 1$  we rescale the limits in (25)

$$\tilde{E}_2^{(\alpha)} = \lim_{K \rightarrow \infty} \sum_{|m| \leq K} \sum_{|n| \leq \lfloor K/\alpha \rfloor} \frac{1}{(m+n\tau)^2}.$$

Informally, this has the “effect” of inscribing the square in the rectangle and motivates computing the contributions from the points in the difference using the plane-wave formulas appropriate for  $\Lambda_{(\bullet,\pm)}^K$ . Arguing as above and using the integral identity (29) we compute

$$\Delta(\alpha, \tau) = -2 \lim_{K \rightarrow \infty} \sum_{n=K+1}^{\lfloor K/\alpha \rfloor} \sum_{m=-K}^{m=K} \int_0^\infty \lambda e^{i\lambda(m+n\tau)} d\lambda$$

$$\begin{aligned}
&= \frac{2}{\tau} \int_0^\infty \frac{(e^{-i\lambda} - e^{i\lambda})(e^{i\lambda\tau} - e^{i\lambda\tau/\alpha})}{\lambda} d\lambda \\
&= -\frac{4i}{\tau} \left( \arctan\left(-\frac{1}{i\tau}\right) - \arctan\left(-\frac{\alpha}{i\tau}\right) \right). \tag{30}
\end{aligned}$$

Although perhaps not obvious at first glance, the formula (30) is the same as the formula for  $\Delta(\alpha, \tau)$  in (27). Using standard trig identities we have

$$\lim_{z' \rightarrow z} \left( \arctan(z) - \arctan\left(-\frac{1}{z'}\right) \right) = \lim_{z' \rightarrow z} \arctan\left(\frac{zz' + 1}{z' - z}\right) = \pm \frac{\pi}{2}. \tag{31}$$

Furthermore, for  $z = i\tau$  or  $z = i\tau/\alpha$  with  $\tau$  satisfying (10) and  $\alpha > 0$ , we find that we should choose the minus sign in (31). Using this identity and algebra we observe that (30) is equal to (27), thus (27) holds for all  $\alpha > 0$ . †

We use this theorem to find the connection between our summation and Eisenstein's. Starting from Eisenstein's summation convention we obtain

$$\begin{aligned}
E_2 &= \lim_{N \rightarrow \infty} \sum_{n=-N}^N \lim_{M \rightarrow \infty} \sum_{m=-M}^M \frac{1}{(m+n\tau)^2} \\
&= \lim_{N \rightarrow \infty} \sum_{n=-N}^N \lim_{\alpha \rightarrow \infty} \sum_{m=-\lfloor \alpha \cdot N \rfloor}^{m=-\lfloor \alpha \cdot N \rfloor} \frac{1}{(m+n\tau)^2} \\
&= \lim_{\alpha \rightarrow \infty} \lim_{N \rightarrow \infty} \sum_{n=-N}^N \sum_{m=-\lfloor \alpha \cdot N \rfloor}^{m=\lfloor \alpha \cdot N \rfloor} \frac{1}{(m+n\tau)^2} \\
&= \tilde{E}_2 - \frac{4i}{\tau} \arctan(i\tau). \tag{32}
\end{aligned}$$

Standard estimates justify commuting the  $N$  and  $\alpha$  limits between the second and third lines, and we used (27) to compute this limit. As  $\tau \neq 0$ , (32) shows that, for finite  $\tau$ , the value of our sum is always different from Eisenstein's. In the limit  $\tau \rightarrow i\infty$ ,  $|\Re(\tau)| \leq 1/2$ , both summations are equal and presumably converge to  $2\zeta_R(2) = \pi^2/3$ .

In a similar vein, we compute the difference between taking Eisenstein's limit and "its reverse". Arguing as above we have that

$$\begin{aligned}
\lim_{M \rightarrow \infty} \sum_{m=-M}^M \lim_{N \rightarrow \infty} \sum_{n=-N}^N \frac{1}{(m+n\tau)^2} &= \tilde{E}_2 + \lim_{\alpha \rightarrow 0} \Delta(\alpha, \tau) \\
&= \tilde{E}_2 - \frac{4i}{\tau} \left( \arctan\left(\frac{i}{\tau}\right) + \frac{\pi}{2} \right) \tag{33}
\end{aligned}$$

Taking the difference between (32) and (33) and we obtain

$$\left( \lim_{N \rightarrow \infty} \sum_{n=-N}^N \lim_{M \rightarrow \infty} \sum_{m=-M}^M - \lim_{M \rightarrow \infty} \sum_{m=-M}^M \lim_{N \rightarrow \infty} \sum_{n=-N}^N \right) \frac{1}{(m+n\tau)^2} = \frac{2\pi i}{\tau}.$$

For a different proof of this fact see Walker, [11].

Finally, in the case of the square lattice,  $\tau = i$ , we observe that  $f_1(i, \lambda) = f_2(i, \lambda)$ . Collecting factors in (21) and performing algebra we find that

$$\tilde{E}_s(i) = e^{-is\pi/4} \cos\left(\frac{\pi}{2}s\right) \cos\left(\frac{\pi}{4}s\right) \frac{8}{\Gamma(s)} \int_0^\infty \lambda^{s-1} e^{-\lambda} f_1(i, \lambda) d\lambda.$$

Corresponding to the added  $\pi/2$ -symmetry of the square lattice, the product of cosines causes the sum to vanish for  $s = n \geq 3$ ,  $n$  not a multiple of four. Similarly, as the representation (21) is valid for  $n = 2$ , we conclude  $\tilde{E}_2(i) = 0$ . Substituting this value into (26), and taking the large  $\alpha$  (small  $\alpha$ ) limits we obtain

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{n=-N}^N \lim_{M \rightarrow \infty} \sum_{m=-M}^M \frac{1}{(m+in)^2} &= \pi \\ \lim_{M \rightarrow \infty} \sum_{m=-M}^M \lim_{N \rightarrow \infty} \sum_{n=-N}^N \frac{1}{(m+in)^2} &= -\pi, \end{aligned}$$

well-known identities in the fast multipole community (see, for example, [5]).

## 4 Weierstrass elliptic functions

Our derivations of integral formulas for Weierstrass' elliptic functions proceed in much the same manner as above. As with the Eisenstein series, we begin with the definitions of the functions as sums over the lattice (7) and (8). Next, we group terms in the sum as in (14), substitute the appropriate plane-wave expansions from lemma 1, and sum the resulting geometric series using the identities of corollary 3.

As a preliminary note, the integral representations for  $\wp(z, \tau)$  and  $\zeta(z, \tau)$  which we derive in theorem 9 are not valid for all  $z \in \mathbf{C}$ , but rather have a finite domain of validity. This is a consequence of the way in which we group terms. More precisely, we require that  $z \in D(\tau)$  defined by

$$D(\tau) = \{z \mid \Re(-1 \pm z \pm \tau) < 0, \Im(\tau \pm z) > 0\}. \quad (34)$$

As  $\tau$  is in the region (10), one may verify that  $D(\tau)$  is an open set containing the origin. However, we point out that  $D(\tau)$  may not contain the fundamental period parallelogram of the lattice. For example, the standard hexagonal lattice has generators  $(1, \tau) = (1, 1/2 + i\sqrt{3}/2)$ . Thus a corner of  $\Lambda_0$  is given by the point  $z_0 = 1/2 + \tau/2 = 3/4 + i\sqrt{3}/4$ . However,  $\Re(-1 + z_0 + \tau) = 1/4 > 0$ , violating the first inequality in (34).

With this aside, we prove the following.

**Theorem 9.** *Assume  $\Lambda = \Lambda(\mu, \nu)$  is an arbitrary complex lattice with generators chosen such that  $\tau = \nu/\mu$  is in the fundamental region (10), and that the complex number  $z$  is in the domain  $D(\tau)$  defined by the inequalities (34). We have the following integral expressions for the inhomogeneous elliptic functions  $\wp(z, \tau)$ , and  $\zeta(z, \tau)$ :*

$$\wp(z, \tau) = \frac{1}{z^2} + 8 \int_0^\infty \lambda \left[ e^{-\lambda} \sinh^2\left(\frac{z\lambda}{2}\right) f_1(\lambda, \tau) + e^{i\tau\lambda} \sin^2\left(\frac{z\lambda}{2}\right) f_2(\lambda, \tau) \right] d\lambda, \quad (35)$$

and,

$$\zeta(z, \tau) = \frac{1}{z} + 4 \int_0^\infty [e^{-\lambda}(z\lambda - \sinh(z\lambda))f_1(\lambda, \tau) - e^{i\tau\lambda}(z\lambda - \sin(z\lambda))f_2(\lambda, \tau)] d\lambda, \quad (36)$$

where the functions  $f_1, f_2$  are defined by (17) and (19). We may evaluate the homogeneous functions,  $\wp(x|\mu, \nu)$  and  $\zeta(x|\mu, \nu)$ , via the appropriate scaling relations (9) and (35)-(36) with the proviso that  $x/\mu = z \in D(\tau)$ .

**Proof:**

In computing the integral representation for the  $\wp$  function we will group terms of the sum (7) as in (14). As in the computation of the Eisenstein sums, we wish to combine the contributions from the sums over  $\Lambda_{(\pm, \bullet)}^K$ . We compute

$$\begin{aligned} \sum_{\omega \in \Lambda_{(-, \bullet)}^K} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right) &= \sum_{m=1}^K \sum_{n=-m}^m \epsilon_{mn} \left( \frac{1}{(z + m + n\tau)^2} - \frac{1}{(m + n\tau)^2} \right) \\ \sum_{\omega \in \Lambda_{(+, \bullet)}^K} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right) &= \sum_{m=1}^K \sum_{n=-m}^m \epsilon_{mn} \left( \frac{1}{(-z + m + n\tau)^2} - \frac{1}{(m + n\tau)^2} \right). \end{aligned}$$

Therefore the contributions from both quadrants may be expressed as a single sum over  $\Lambda_{(+, \bullet)}^K$  of a modified summand. Furthermore, under the assumption  $z \in D$ , all of the poles in this sum may be expressed using the  $\Re(\omega) > 0$  plane-wave expansion from (12).

$$\begin{aligned} \sum_{\omega \in \Lambda_{(\pm, \bullet)}^K} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right) &= \sum_{m=1}^K \sum_{n=-m}^m \epsilon_{mn} \left( \frac{1}{(-z + m + n\tau)^2} - \frac{2}{(m + n\tau)^2} + \frac{1}{(z + m + n\tau)^2} \right) \\ &= \int_0^\infty \lambda (e^{\lambda z} - 2 + e^{-\lambda z}) \left( \sum_{m=1}^K \sum_{n=-m}^m \epsilon_{mn} e^{-\lambda(m + n\tau)} \right) d\lambda \\ &= 8 \int_0^\infty \lambda \sinh^2 \left( \frac{z\lambda}{2} \right) \left( e^{-\lambda} f_1(\tau, \lambda) - e^{-\lambda(K+1)} f_1^{(K)}(\tau, \lambda) \right) d\lambda. \quad (37) \end{aligned}$$

Arguing as before, we find the large  $K$  limit of the  $K$ -dependent term to be zero. We compute the contribution from the terms in the quadrants  $\Lambda_{(\bullet, \pm)}^K$  in an analogous manner. Adding this result to (37) gives (35).

The derivation of the expression for the  $\zeta$  function is similar. In brief, the sum over the quadrants  $\Lambda_{(\pm, \bullet)}^K$  may again be expressed as a sum over the single quadrant  $\Lambda_{(+, \bullet)}^K$  in which we substitute the appropriate plane-wave expansion. Thus,

$$\begin{aligned} \sum_{\omega \in \Lambda_{(\pm, \bullet)}^K} \left( \frac{1}{(z - \omega)} + \frac{1}{\omega} + \frac{z}{\omega^2} \right) &= \sum_{m=1}^K \sum_{n=-m}^m \epsilon_{mn} \left( -\frac{1}{-z + m + n\tau} + \frac{2z}{(m + n\tau)^2} + \frac{1}{z + m + n\tau} \right) \\ &= \int_0^\infty (e^{\lambda z} + 2z\lambda - e^{-\lambda z}) \left( \sum_{m=1}^K \sum_{n=-m}^m \epsilon_{mn} e^{-\lambda(m + n\tau)} \right) d\lambda \end{aligned}$$



$$= 4 \int_0^\infty (z\lambda - \sinh(z\lambda)) \left( e^{-\lambda} f_1(\lambda, \tau) - e^{-\lambda(K+1)} f_1^{(K)}(\lambda, \tau) \right) d\lambda. \quad (38)$$

As before the  $K$ -dependent term goes to zero in the limit. The analogous sums over  $\Lambda_{(\bullet, \pm)}^K$  give the other half of the expression (36).  $\dagger$

Remarks:

1. As an alternative to the above derivation of theorem 9, we recall that the Eisenstein series appear as coefficients in the Laurent expansion for Weierstrass'  $\wp$  function

$$\begin{aligned} \wp(z) &= \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right) \\ &= \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \sum_{n=3}^{\infty} (n-1) z^{n-2} \frac{1}{\omega^n} \\ &= \frac{1}{z^2} + \sum_{n=3}^{\infty} (n-1) E_n z^{n-2}. \end{aligned}$$

Substituting the integral representations (21) for the  $E_n$ , the Taylor series may be summed explicitly inside the integrand. The formula (35) above follows after algebraic simplification. Furthermore, the expression (36) for the  $\zeta$  function follows from anti-differentiation of (35).

2. As with the series  $\tilde{E}_2$ , and its dependence on aspect ratios derived in theorem 8, the slowly decaying terms of the sums defining  $\wp$  and  $\zeta$  manifest themselves at the origin in the integral representations (equations (7), (8) and (35), (36) respectively). In the integral representations, we observe that Weierstrass' "correction" terms are arranged in such a way as to create third order zeros at  $\lambda = 0$ , which appropriately balance the second order poles from  $f_1$  and  $f_2$ .

## 5 Conclusion

We conclude with a brief discussion of our results in relation to previous research in this field. To the best of our knowledge, there is no analog to the integral expressions for the  $\wp$  and  $\zeta$  functions (35) and (36). The possibility of developing numerical routines for evaluation of these functions based on these representations deserves further study. We observe that the integrands are not extremely oscillatory, and decay exponentially. Thus  $N$ -point Gauss-Laguerre quadrature rules will converge rapidly in  $N$ . As one drawback, there is the perhaps awkward domain of validity in  $z$ . However, it may be that symmetries of the  $\wp$  and  $\zeta$  functions imply that it is sufficient to evaluate them over domains that are smaller than the fundamental period parallelogram. Furthermore, at least for the  $\wp$  function, there exists the following closed-form Fourier-like expansion [10]

$$\wp(z, \tau) = -2 \left( \frac{1}{6} + \sum_{n=1}^{\infty} \frac{1}{\sin^2(n\pi\tau)} \right) + \frac{\pi^2}{\sin^2(\pi z)} - 8\pi^2 \left( \sum_{n=1}^{\infty} \frac{ne^{2\pi i \tau n}}{1 - e^{2\pi i \tau n}} \cos(2\pi n z) \right). \quad (39)$$

Both summands in (39) are exponentially decreasing and the sums converge rapidly—stiff competition from a numerical perspective. Nevertheless, we have not fully explored the relative merits of this approach over the plane-wave representation (35). In addition, the integral representations may have further analytic implications.

Turning to the representations for the Eisenstein series, the existence of  $\tilde{E}_2$  (corollary 5) is not unexpected. In addition to the original finiteness proofs given by Eisenstein, many years prior to this present work, Walker derived the remarkable formula for the conditionally convergent series (see [10])

$$\lim_{K \rightarrow \infty} \sum_{0 < m^2 + n^2 \leq K^2} \frac{1}{(m + n\tau)^2} = \frac{-2\pi}{1 - i\tau} - 4\pi i \frac{\eta'(\tau)}{\eta(\tau)},$$

where the Dedekind  $\eta(\tau)$ -function with  $\Im(\tau) > 0$  is defined by

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i \tau n}).$$

We also note that a different treatment, initiated by Hecke, has become a standard approach to resolving convergence and transformation properties of  $E_2$  [9].

As indicated by theorem 6, our expressions are quite general, and have broad implications. We note that Riemann demonstrated both the functional equation satisfied by  $\zeta_R(s)$ , and the evaluation of  $\zeta_R(-2n + 1)$  (and, via the functional equation,  $\zeta_R(2n)$ ) in terms of Bernoulli numbers using the “version” of theorem 6 appropriate for his zeta function. Similarly, we anticipate that a residue argument will give the evaluation of  $\tilde{E}_n = E_n$  in terms of multiple Bernoulli numbers. For an alternative treatment of Eisenstein series for negative even integers using Hecke convergence factors see the recent work of Pribitkin [8]. The functional equation satisfied by the continuation of  $\tilde{E}_s$  is more elusive. We are currently pursuing this and hope to report our results in the future.

Finally, there is a possibility that representations of the form (21) may exist for certain Dirichlet series

$$G(s, \chi) = \sum_{\omega \in \Lambda \setminus \{0\}} \frac{\chi(\omega)}{|\omega|^{2s}}$$

$$\chi(\omega) = \exp(i(m\mu\alpha + n\nu\beta)),$$

for  $\alpha, \beta \in \mathbf{R}$ . (These are called “Kronecker series” in [12], Chapter VIII.) A detailed discussion of the convergence of these series is given in [2]. We note that Laplace-Mellin techniques have been employed frequently in the analysis of such series. The approach up until now has been to think of

$$|\omega_{m,n}|^2 = |m\mu + n\nu|^2 = Q(m, n)$$

as defining a positive definite quadratic form taking  $m$  and  $n$  as arguments. Treating this form as “indivisible”, one may use the  $\Re(\omega) > 0$  plane-wave formula in lemma 1 and obtain the integral representation

$$\sum_{\omega \in \Lambda \setminus \{0\}} \frac{\chi(\omega)}{|\omega|^{2s}} = \frac{1}{\Gamma(s)} \int_0^\infty \lambda^{s-1} \sum_{\omega \in \Lambda \setminus \{0\}} \chi(\omega) e^{-\lambda Q(m,n)} d\lambda.$$

The analysis then proceeds via  $\theta$  functions.

Our approach would be different. “Plane-wave-like” representations exist for the function  $f(x, y) = \sqrt{x^2 + y^2}$ . Formally, one may take the true plane-wave expressions for  $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$  derived in [3], and set  $z = 0$ . The result is a 2D integral — as opposed to the Eisenstein case analyzed above where one complex dimension (two real) collapses into a 1D integral. However, the critical element of this representation is that the exponential function in the “plane-wave” representation is linear in  $m$  and  $n$ . Again, as a consequence, we observe that the summation under the integrand becomes a 2D geometric series. We are considering this as a possible direction for future research.

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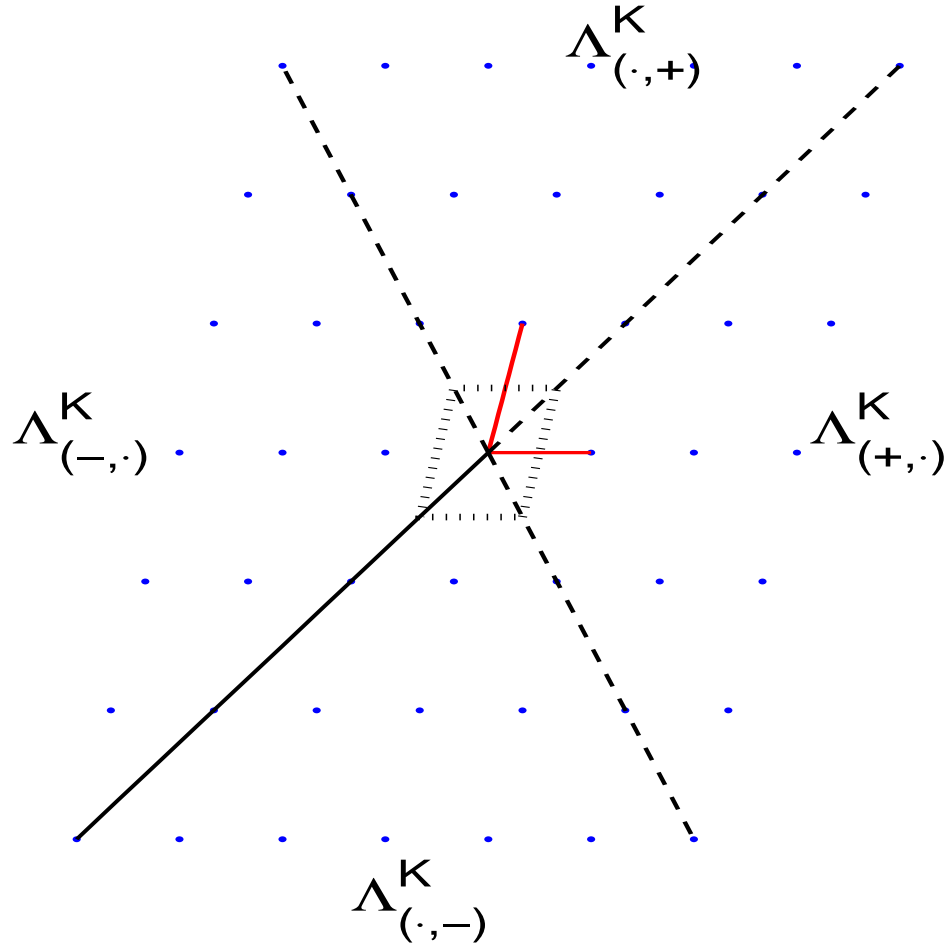


Figure 1: Partition of  $\Lambda^K$  into subregions. The generators  $(1, \tau)$  are shown in red. The central dotted region is the boundary of the fundamental domain. The dashed lines show the divisions into  $\Lambda^K_{(\pm, \pm)}$ . The solid black line is the branch cut.

## References

- [1] L. V. Ahlfors. *Complex Analysis*. McGraw-Hill, Inc., 1979.
- [2] D. Borwein, J. M. Borwein, and C. Pinner. Convergence of Madelung-like lattice sums. *Transactions of the American Mathematical Society*, 350(8):3131–3167, Aug. 1998.
- [3] H. Cheng, L. Greengard, and V. Rokhlin. A fast adaptive multipole algorithm in three dimensions. *Journal of Computational Physics*, 155(2):468–498, Nov. 1999.
- [4] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Academic Press Inc., fifth edition, 1994.
- [5] L. Greengard. *The rapid evaluation of potential fields in particle systems*. ACM distinguished dissertations. MIT Press, 1988.
- [6] T. Hrycak and V. Rokhlin. An improved fast multipole algorithm for potential fields. *SIAM Journal of Scientific and Statistical Computing*, 19(6):1804–1826, Nov. 1998.
- [7] J. Huang. Integral representations of harmonic lattice sums. *Journal of Mathematical Physics*, 40(10):5240–5246, Oct. 1999.
- [8] W. d. A. Pribitkin. Eisenstein series and Eichler integrals. In *Analysis, Geometry, Number Theory: The Mathematics of Leon Ehrenpreis*, volume 251 of *Contemporary Mathematics*, pages 463–468. American Mathematical Society, 2000.
- [9] B. Schoeneberg. *Elliptic Modular Functions*. Springer-Verlag, 1974.
- [10] P. Walker. On the functional equations satisfied by modular functions. *Mathematika*, 25:185–190, 1978.
- [11] P. Walker. *Elliptic Functions: A Constructive Approach*. John Wiley and Sons, 1996.
- [12] A. Weil. *Elliptic Functions According to Eisenstein and Kronecker*, volume 88 of *A Series of Modern Surveys in Mathematics*. Springer-Verlag, 1976.